An intrinsic proof of Gromoll-Grove diameter rigidity theorem

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Dedicated to Professor Karsten Grove on his sixtieth birthday

1 Introduction

We will present a new proof of the following Gromoll-Grove diameter rigidity theorem.

Theorem A *Let M^n be a simply connected Riemannian manifold with sectional curvature $K \geq 1$. Suppose that $\mathrm{Diam}(M^n) = \frac{\pi}{2}$ and M^n is not homeomorphic to a sphere S^n. Then M^n is isometric to one of $\mathbb{C}P^{\frac{n}{2}}$, $\mathbb{H}P^{\frac{n}{4}}$ or $\mathbb{C}aP2$, i.e., M^n is isometric to a projective symmetric space over complex numbers, or quaternion numbers or Calay numbers.*

Our proof does not use any loop spaces, which is totally different from [Wil]. Among other things, we use the Hessian comparison theorem for distance functions and the spherical metric on the tangent space instead, see Section 3 below. Although our new proof is longer than its earlier version, the most of arguments below remain to be elementary and self-contained.

2 The Gromoll-Grove fibration

We need to recall some known results from [GG1], in order to complete the proof of Theorem A. The results of [GG1] are related to the following example.

Example 2.0. (1) Let $M^n = \mathbb{C}P^{\frac{n}{2}}$ with the classical Fubini-Study metric and diameter $\frac{\pi}{2}$. Let $B_r(p)$ be the metric ball of radius r and center p in $\mathbb{C}P^{\frac{n}{2}}$, and let $S_r(p) = \partial B_r(p)$ be the metric sphere of radius r centered at p. It is well-known that $S_{\frac{\pi}{2}}(p)$ is isometric to $\mathbb{C}P^{\frac{n}{2}-1}$.

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For each $p \in \mathbb{C}P^{\frac{n}{2}}$, we consider a polar coordinate $\{(r, \Theta)\}$ of the tangent space $T_p(\mathbb{C}P^{\frac{n}{2}})$ and the exponential map $\mathrm{Exp}_p : T_p(\mathbb{C}P^{\frac{n}{2}}) \to \mathbb{C}P^{\frac{n}{2}}$.

Let us choose a spherical metric

$$g_1 = dr^2 + (\sin r)^2 d\Theta^2$$

on the set $B_\pi(0) = \{(r, \Theta)|0 \le r < \pi, \Theta \in S^{n-1}\}$, where $d\Theta^2$ is the canonical metric of constant curvature 1 on the unit sphere S^{n-1}. With respect to the spherical metric g_1, the exponential map

$$\mathrm{Exp}_p : \quad \begin{aligned} S_{\frac{\pi}{2}}(0) &\rightarrow S_{\frac{\pi}{2}}(p) \\ \tfrac{\pi}{2}\Theta &\rightarrow \mathrm{Exp}_p(\tfrac{\pi}{2}\Theta) \end{aligned}$$

is a Hopf fibration.

Furthermore, for each $q \in S_{\frac{\pi}{2}}(p)$, the fiber $\mathrm{Exp}_p^{-1}(q)$ is a great circle in the equator $(S_{\frac{\pi}{2}}(0), g_1)$ of the unit sphere $S^n = (\bar{B}_\pi(0), g_1)$.

(2) We are going to elaborate the above construction by replacing the point p by a totally geodesic submanifold $\mathbb{C}P^m \subset \mathbb{C}P^{\frac{n}{2}}$ with $1 \le m < \frac{n}{2}-1$, for the case $\frac{n}{2} \ge 3$. We let $U_r(\mathbb{C}P^m) = \{z \in \mathbb{C}P^{\frac{n}{2}} \,|\, d(z, \mathbb{C}P^m) < r\}$ be the tubular neighborhood and $\partial[U_r(\mathbb{C}P^m)]$ its boundary.

Then $\partial[U_{\frac{\pi}{2}}(\mathbb{C}P^m)]$ is isometric to a totally geodesic $\mathbb{C}P^{m'} \subset \mathbb{C}P^{\frac{n}{2}}$ with $m' = \frac{n}{2} - m - 1$. In this case, for each pair $p \in \mathbb{C}P^m$ and $q \in \mathbb{C}P^{m'}$ with distance $d(p, q) = \frac{\pi}{2}$, we still have that the fiber $\mathrm{Exp}_p^{-1}(q)$ is a great circle in the equator $(S_{\frac{\pi}{2}}(0), g_1)$ of the unit sphere $S^n = (B_\pi(0), g_1)$, where $\bar{B}_r(0) \subset T_p(\mathbb{C}P^{\frac{n}{2}})$.

In fact, $\mathbb{C}^{\frac{n}{2}+1}$ has a decomposition $\mathbb{C}^{\frac{n}{2}+1} = \mathbb{C}^{m+1} \times \mathbb{C}^{m'+1}$. Such a decomposition induces a spherical join of S^{2m+1} and $S^{2m'+1}$. More precisely, for each unit vector $\vec{u} \in S^{n+1} \subset \mathbb{C}^{\frac{n}{2}+1}$, there are $\vec{v} \in S^{2m+1}$ and $\vec{w} \in S^{2m'+1}$

$$\vec{u} = (\cos r)\vec{v} + (\sin r)\vec{w}$$

for some $r \in [0, \frac{\pi}{2}]$. One can write $S^{n+1} = S^{2m+1} \star S^{2m'+1}$, where $\frac{n}{2} = m + m' + 1$. It follows that $\mathbb{C}P^{\frac{n}{2}}$ can be viewed as the *"projective join"* of $\mathbb{C}P^m$ and $\mathbb{C}P^{m'}$.

The pair of sub-manifolds $\{\mathbb{C}P^m, \mathbb{C}P^{m'}\}$ with $d(\mathbb{C}P^m, \mathbb{C}P^{m'}) = \frac{\pi}{2}$ above is called a *dual pair* of convex subsets of $\mathbb{C}P^{\frac{n}{2}}$ in [GG1].

(3) When M^n is isometric to either $\mathbb{H}P^{\frac{n}{4}}$ or $\mathbb{C}aP^2$, there are similar decompositions. Q.E.D.

Inspired by Example 2.0, we consider the convexity of subset $[M - B_r(p)]$, without the assumption $\mathrm{Diam}(M) = \frac{\pi}{2}$. Let $\mathrm{Inj}_M(x)$ denote the injectivity radius of M at x.

Proposition 2.1 *Let M be a complete smooth Riemannian manifold with sectional curvature ≥ 1 and $\mathrm{Diam}(M) \ge \frac{\pi}{2}$. Suppose that $\sigma : [0, \ell] \to M$ is a length-minimizing geodesic of unit speed from x. Then, for any $0 < r < \ell$, the second fundamental form of $S_r(x)$ at $\sigma(r)$ with*

respect to the normal vector $\sigma'(r)$ is less than or equal to $\cot(r)I$ at $\sigma(r)$ in the barrier sense, where I is the identity matrix.

Consequently, if $Inj_M(x) \geq \frac{\pi}{2}$, then $[M - B_{\frac{\pi}{2}}(x)]$ is a convex subset of M. In addition, if $\mathrm{Diam}(M) \geq \ell > \frac{\pi}{2}$, then $[M - B_\ell(x)]$ is strictly convex.

Proof. This is a direct consequence of the Hessian comparison (see [Pe, p145]) for the distance function.   Q.E.D.

Proposition 2.2 ([GG1]) *Let M be a complete smooth Riemannian manifold with sectional curvature ≥ 1 and $\mathrm{Diam}(M) = \frac{\pi}{2}$. If M is simply-connected and if M has the integral cohomology ring of either $\mathbb{C}P^2$, $\mathbb{H}P^2$ or the Cayley plane $\mathbb{C}aP^2$, then there exists at least one point $p \in M$ with injectivity radius $\mathrm{Inj}_M(p) \geq \frac{\pi}{2}$.*

When $d(p, q) = \mathrm{Diam}(M) = \frac{\pi}{2}$, the subset $S_{\frac{\pi}{2}}(p)$ is a critical sub-manifold of the distance function $f(x) = d(x, p)$. If $\mathrm{Inj}_M(p) \geq \frac{\pi}{2}$, by Proposition 2.2 above, $S_{\frac{\pi}{2}}(p)$ is a totally geodesic submanifold. In fact, the dual convex subset $S_{\frac{\pi}{2}}(p)$ has some extra properties (cf. [GG1]), which we recall in the sequel.

Following [GG1], for $A \subset M$ we let

$$A' = \{y \in M \mid d(y, A) = \frac{\pi}{2}\}.$$

The following result was also stated in [GG1].

Proposition 2.3 ([GG1]) *Let M be a simply connected Riemannian manifold with sectional curvature ≥ 1 and $\mathrm{Diam}(M) = \frac{\pi}{2}$. Suppose that the injectivity radius $\mathrm{Inj}_M(p)$ of M at p is equal to $\frac{\pi}{2}$ and that M is not homeomorphic to a sphere. Then*

(1) M has integral cohomology ring of either $\mathbb{C}P^{\frac{n}{2}}$, $\mathbb{H}P^{\frac{n}{4}}$ or the Cayley plane $\mathbb{C}aP^2$;

(2) if $A = \{p\}$, then $A' = \{y \mid d(y, p) = \frac{\pi}{2}\}$ is a closed totally geodesic submanifold of positive dimension;

(3) if $A = \{p\}$, then $(A')' = A$ and the cut radius $\mathrm{Cut}_M(A')$ is equal to $\frac{\pi}{2}$ as well;

(4) if $S_p(M) = \{\vec{v} \in T_pM \mid \|\vec{v}\| = 1\}$ then

$$\pi_p = \widetilde{\mathrm{Exp}_p} : \quad S_p(M) \quad \rightarrow \quad A'$$
$$\vec{v} \quad \rightarrow \quad \mathrm{Exp}_p(\tfrac{\pi}{2}\vec{v})$$

is a Riemannian submersion;

(5) if the Riemannian submersion $\pi_p : S_p(M) \rightarrow A'$ is a great circle fibration, then M is isometric to either $\mathbb{C}P^{\frac{n}{2}}$, $\mathbb{H}P^{\frac{n}{4}}$ or the Cayley plane $\mathbb{C}aP^2$.

Notice that $\{q\}''' = \{q\}'$ holds for all $q \in M$, when $\mathrm{Diam}(M) = \frac{\pi}{2}$. We choose $A' = \{q\}'$ and $A = A''$. It is possible that $\min\{\dim A, \dim A'\} > 0$, see Example 2.0 above. If M is allowed

to be non-simply-connected, and if M^3 is a lens space, then $\min\{\dim A, \dim A'\} = 1$. In other words, it might be difficult to find a point p with $\mathrm{Inj}_M(p) = \frac{\pi}{2}$ when $\mathrm{Diam}(M) = \frac{\pi}{2}$. We can not choose A with $\dim A = 0$ at the first place.

Thus, we need to describe the remaining case of $\min\{\dim A, \dim A'\} > 0$, where $A'' = A$ and $\{A, A'\}$ is a pair of dual convex subsets. It was shown in [GG1] that both A and A' are connected totally geodesic submanifolds without boundaries.

In what follows, we always let

$$S_q^\perp(B, M) = \{\vec{v} \in T_p M \mid \vec{v} \perp T_q(B), |\vec{v}| = 1\}$$

be the unit normal bundle of B in M, when A' is a submanifold of M.

Proposition 2.4 *([GG1]) Let M be a simply-connected Riemannian manifold with sectional curvature ≥ 1 and diameter $\mathrm{Diam}(M^n) = \frac{\pi}{2}$. For any $z \in M$ with $S_{\frac{\pi}{2}}(z) \neq \emptyset$, we let $A' = S_{\frac{\pi}{2}}(z)$ and $A = A''$. Suppose that M^n is not homeomorphic to S^n. Then*
  (1) both A and A' are simply-connected;
  (2) the cut radius $\mathrm{Cut}_M(A)$ of A in M is equal to $\frac{\pi}{2}$ and $\mathrm{Cut}(A) = A'$;
  (3) the cut radius $\mathrm{Cut}_M(A')$ of A' in M is equal to $\frac{\pi}{2}$ and $\mathrm{Cut}(A') = A$;
  (4) if $\dim(A') > 0$, then

$$\pi_p = \widetilde{\mathrm{Exp}}_p: \quad S_p^\perp(A, M) \quad \to \quad A'$$
$$\vec{v} \quad \to \quad \mathrm{Exp}_p(\tfrac{\pi}{2}\vec{v})$$

is a Riemannian submersion; similarly, if $\dim A > 0$ then $\pi_q : S_q^\perp(A', M) \to A$ is a Riemannian submersion for all $q \in A'$; furthermore, $\dim[\pi_p^{-1}(q)]$ is equal to one of $\{1, 3, 7\}$; $\dim M$, $\dim A$ and $\dim A'$ are even integers;

  (5) if the Riemannian submersion $\pi_p : S_p^\perp(A, M) \to A'$ with $\dim A' > 0$ is a great circle fibration for all $p \in A$ and if the Riemannian submersion $\pi_q : S_q^\perp(A', M) \to A$ is a great circle fibration for all $q \in A'$ whenever $\dim A > 0$, then M^n is isometric to one of symmetric spaces $\mathbb{C}P^{\frac{n}{2}}$, $\mathbb{H}P^{\frac{n}{4}}$ or $\mathbb{C}aP^2$.

Definition 2.5 *Let M^n be a simply connected Riemannian manifold with sectional curvature ≥ 1. Suppose that $\mathrm{Diam}(M^n) = \frac{\pi}{2}$, $A'' = A$ and $(p, q) \in A \times A'$. When $\dim A' > 0$, the Riemannian submersion*
$$\pi_p: \quad S_p^\perp(A, M) \quad \to \quad A'$$
$$\vec{v} \quad \to \quad \mathrm{Exp}_p(\tfrac{\pi}{2}\vec{v})$$

is called the Gromoll-Grove fibration with the total space $S_p^\perp(A, M)$.

Similarly, when $\dim A > 0$, the fibration $\pi_q : S_q^\perp(A', M) \to A$ is called the Gromoll-Grove fibration as well.

In next section, we will show that the Gromoll-Grove fibration

$$S^k \to S_p^\perp(A, M) \to A'$$

is a great circle fibration for some $k \in \{1, 3, 7\}$ whenever $\dim(A') > 0$; and hence M must be isometric to a symmetric space by [Ran].

3 The Gromoll-Grove fibration is isometrically congruent to a Hopf fibration

In this section, we will use a new method to show that the Gromoll-Grove fibration is isometrically congruent to a great circle fibration.

Throughout this section, the origin of $T_pM \approx \mathbb{R}^n$ is denoted by 0_p. We will always use a spherical metric g_1 on a ball $B_\pi(0_p) \subset T_pM$:

$$g_1 = dr^2 + (\sin r)^2 d\Theta^2$$

where $\{(r, \Theta)\}$ is the polar coordinate system of $T_pM \approx \mathbb{R}^n$.

We consider the possibly tear-drop shaped fibres in the manifold M, see Section 2 above. For each pair of $p \in A$ and $q \in A'$, we let

$$\Sigma_{p,q} = \{\mathrm{Exp}_p(t\vec{v}) \,|\, \vec{v} \in \pi_p^{-1}(q), 0 \leq t \leq \frac{\pi}{2}\}.$$

and

$$\tilde{\Sigma}_{p,q} = \{\vec{w} \in \mathrm{Exp}_p^{-1}(\Sigma_{p,q}) \,|\, \|\vec{w}\| \leq \frac{\pi}{2}\}$$

be the truncated tangential cone of $\Sigma_{p,q}$ at p.

Our goal is to show that $\tilde{\Sigma}_{p,q}$ is totally geodesic in $(B_\pi^\perp(0_p), g_1) \subset S^n$ and hence $\partial[\tilde{\Sigma}_{p,q}]$ is totally geodesic in S^n. Consequently, $\pi_p^{-1}(q)$ is a k-dimensional circle in S^{n-1}, where k is one of $\{1, 3, 7\}$.

There are three elementary steps to show that $\pi_p^{-1}(q)$ is a k-dimensional circle in S^{n-1}.

Step 1. We will show that *"if $\tilde{\Sigma}_{p,q}$ has the first focal radius $\geq \frac{\pi}{2}$ in S^n at all $z \in \tilde{\Sigma}_{p,q}$ with $0 < |z| < \frac{\pi}{2}$, then $\tilde{\Sigma}_{p,q}$ is a smooth totally geodesic submanifold of S^n".*

Step 2. We will make the following elementary observation. *Suppose contrary, $\tilde{\Sigma}_{p,q}$ had the first focal radius $0 < t_0 < \frac{\pi}{2}$ in $(B_\pi^\perp(0_p), g_1) \subset S^n$ at some $z \in \tilde{\Sigma}_{p,q}$ with $0 < |z| < \frac{\pi}{2}$. Then there would be a Jacobi field $\{J(t)\}$ along a normal geodesic $\sigma_{z,\vec{h}}(t) = \mathrm{Exp}_z^{S^n}(t\vec{h})$ such that $\vec{h} \perp T_z(\tilde{\Sigma}_{p,q})$, $|\vec{h}| = 1$ and $J'(0) \in T_z(\tilde{\Sigma}_{p,q})$.*

Thus, we consider a special class of Jacobi fields with *extra* initial conditions on $J'(0)$:

$$\Gamma_{\sigma_{z,\vec{h}}, \tilde{\Sigma}_{p,q}} = \{J \,|\, J'' + R(\sigma', J)\sigma' = 0, \langle J'(0), X \rangle = -\langle \vec{h}, \nabla_{J(0)} X \rangle, \text{for all } X \in T_z(\tilde{\Sigma}_{p,q})\}$$

5

and

$$\Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}_{p,q}} = \{J \in \Gamma_{\sigma_{z,\vec{h}},\tilde{\Sigma}_{p,q}}, |J'(0) \in T_z(\tilde{\Sigma}_{p,q})\}. \tag{3.1}$$

It will be shown

$$\dim[\Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}_{p,q}}] = \dim[\tilde{\Sigma}] = k+1.$$

This step is applicable to all $(k+1)$-dimensional submanifold $\tilde{\Sigma} \subset S^n$, which is elementary.

Step 3. In this final step, we use Hessian comparison theorem to show that, *"if $\pi_p : S_p^\perp(A, M) \to A'$ is a Riemannian submersion, then, for all non-trivial Jacobi field $J \in \Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}_{p,q}}$, we have $J(t) \neq 0$ for all $t \in (0, \frac{\pi}{2})$."* It follows that $\tilde{\Sigma}_{p,q}$ has the first focal radius $\geq \frac{\pi}{2}$ and hence totally geodesic in S^n. This completes the proof of Grove-Gromoll diameter rigidity Theorem.

Here are the details for each step.

Step 1. We present a sufficient condition for totally geodesic property.

A subset $C \subset M$ is *a*-convex in M if, for all geodesic segments $\sigma : [0, \ell] \to M$ of length $\ell < a$ with endpoints in C, one has $\sigma([0, \ell]) \subset C$.

Proposition 3.1 *If $\tilde{\Sigma}_{p,q}$ has the first focal radius $\geq \frac{\pi}{2}$ in $S^n = (B_\pi(0_p), g_1)$ at all $z \in \tilde{\Sigma}_{p,q}$ with $0 < |z| < \frac{\pi}{2}$ where $B_\pi(0_p) \subset T_p(M^n)$, then*

(1) $\tilde{\Sigma}_{p,q}$ is a smooth totally geodesic submanifold with boundary in $S^n = (B_\pi(0_p), g_1)$; Moreover, $\pi_p^{-1}(q) \approx [\partial \tilde{\Sigma}_{p,q}]$ is a totally geodesic great k-dimensional circle in S^n.

(2) The injectivity radius $\mathrm{Inj}_M(q)$ of q in M is equal to $\frac{\pi}{2}$ for all $q \in A'$.

Proof. (1) Let $\vec{h}_0 \in S^\perp(\tilde{\Sigma}_{p,q}, S^n)$ be a unit norm vector of $\tilde{\Sigma}_{p,q}$ at z_0 and $\sigma_0(t) = \mathrm{Exp}_{z_0}(t\vec{h}_0)$. Let $\mathrm{Exp} : S^\perp(\tilde{\Sigma}_{p,q}, S^n) \times [0, \infty) \to S^n$ be the exponential map along the normal bundle near $(z_0, \vec{h}_0; t)$ for $t \geq 0$. Suppose that $\zeta : (-\delta, \delta) \to S^\perp(\tilde{\Sigma}_{p,q}, S^n)$ is a curve with $\zeta(0) = (z_0, \vec{h}_0)$ and $\zeta(s) = (z(s), \vec{h}(s))$. Then $F(t, s) = \mathrm{Exp}_{z(s)}[t\vec{h}(s)]$ gives rise to a Jacobi field $\{J(t)\}$ defined by

$$J(t) = \frac{\partial F}{\partial s}(t, 0)$$

along σ_0.

Our goal is to show that, under our assumption, we have

$$\langle J(0), \nabla_{J(0)}\vec{h}(s) \rangle = \langle J(0), J'(0) \rangle \geq 0. \tag{3.2}$$

Since $\tilde{\Sigma}_{p,q}$ is not a hypersurface, we consider a tubular neighborhood of $\tilde{\Sigma}_{p,q}$. Choose ε_1 sufficiently small so that $B_{\varepsilon_1}(z_0) \cap \tilde{\Sigma}_{p,q}$ is an embedded $(k+1)$-dimensional ball. Let ε_0 be the

cut-radius of $\tilde{\Sigma}_{p,q} \cap B_{\varepsilon_1}(z_0)$. Choose $\varepsilon < \frac{1}{8} \min\{\varepsilon_0, \varepsilon_1\}$. Then there is a nearest point projection from $B_\varepsilon(z_0) \to \tilde{\Sigma}_{p,q}$. If $\{G(., s)\}$ is an 1-family variation of $\sigma_0|_{[\varepsilon, \ell]}$, which are orthogonal to $\partial[U_\varepsilon(\tilde{\Sigma}_{p,q})]$, then by using the nearest point projection, such a family $\{G(., s)\}$ can be extended as an 1-family of normal geodesics $\{F(., s)\}$ from $\tilde{\Sigma}_{p,q}$ with $F(., 0) = \sigma_0(.)$.

Hence we see that the hypersurface $\partial[U_\varepsilon(\tilde{\Sigma}_{p,q})]$ has focal radius $\geq \frac{\pi}{2} - \varepsilon$ along σ_0, where $U_\varepsilon(C) = \{y \in S^n \,|\, d(y, C) < \varepsilon\}$.

To prove (3.2), it is sufficient to

$$\frac{\langle J(\varepsilon), J'(\varepsilon) \rangle}{|J(\varepsilon)|^2} \geq -\tan\varepsilon. \tag{3.3}$$

Let $\lambda(\varepsilon)$ be an eigenvalues of $II_\varepsilon(X, Y) = -\langle \nabla_X Y, -\sigma_0'(\varepsilon) \rangle$. For (3.3), it is sufficient to show that $\lambda(\varepsilon) \geq -\tan\varepsilon$.

We may isometrically embed S^n into $\mathbb{R}^{n+1}$ as $S^n \approx \{\vec{w} \,|\, \vec{w} \in \mathbb{R}^{n+1}, |\vec{w}| = 1\}$ and identify $0_p \in T_p(M^n)$ with the North pole $e_{n+1} = (0, ..., 0, 1)$. In the unit sphere $S^n \subset \mathbb{R}^{n+1}$, any geodesic of unit speed can be written as $\sigma(t) = (\cos t)\sigma(0) + (\sin t)\sigma'(0)$. Thus, any Jacobi field can be expressed as $J(t) = (\cos t)J(0) + (\sin t)J'(0)$. If $J'(\varepsilon) = \lambda(\varepsilon)J(\varepsilon)$, then we have $J(t) = [\cos(t-\varepsilon) + \lambda(\varepsilon)\sin(t-\varepsilon)]J(\varepsilon)$. The equality $J(t_0) = 0$ holds if and only if $t_0 = \cot^{-1}[-\lambda(\varepsilon)] + \varepsilon$. By our assumption, $t_0 \geq \frac{\pi}{2}$. It follows that

$$-\lambda(\varepsilon) = \cot[t_0 - \varepsilon] \leq \cot[\frac{\pi}{2} - \varepsilon] = \tan\varepsilon$$

This completes the proof of (3.2) and (3.3).

Hence, we showed that $\tilde{\Sigma}_{p,q}$ is totally geodesic at z with $0 < d(z, 0_p) < \frac{\pi}{2}$. It remains to show that $\partial[\tilde{\Sigma}_{p,q}]$ is a k-dimensional great circle in S^n.

For this purpose, we let $S^{n-1} = \{(\vec{v}, 0) \,|\, (\vec{v}, 0) \in S^n \subset \mathbb{R}^{n+1}\}$ be the equator of S^n. Let $\Psi : S^n - \{\pm e_{n+1}\} \to S^{n-1}$ be the nearest point projection to the equator S^{n-1} given by $\Psi(z) = \frac{z - \langle z, e_{n+1} \rangle e_{n+1}}{|z - \langle z, e_{n+1} \rangle e_{n+1}|}$. It is easy to see that Ψ takes a geodesic segment in S^n to a arc of a great circle in S^{n-1}. Since $\tilde{\Sigma}_{p,q}$ is totally geodesic at z with $0 < d(z, 0_p) < \frac{\pi}{2}$, $\Psi(\tilde{\Sigma}_{p,q})$ must be contained in a totaly geodesic subset in the equator S^{n-1}. However, it is easy to see that $\Psi(\tilde{\Sigma}_{p,q}) = \partial[\tilde{\Sigma}_{p,q}]$. It follows that $\partial[\tilde{\Sigma}_{p,q}]$ is totally geodesic in S^n. Consequently, $\pi_p^{-1}(q)$ is a great circle in $S^{n-1} \subset \mathbb{R}^n$.

(2) We first observe $\text{Inj}_{A'}(q) = \frac{\pi}{2}$, due to [Ran]. Here is a direct proof of $\text{Inj}_{A'}(q) = \frac{\pi}{2}$ without using results of [Ran].

We now consider the Riemannian submersion $\pi_p : S_p^\perp(A, M) \to A'$, where $S^{n-1} = (\partial B_{\frac{\pi}{2}}(0), g_1)$ is the equator of $S^n \subset \mathbb{R}^{n+1}$. Since $\pi^{-1}(q)$ is totally geodesic, it is a great circle. We still isometrically embed S^n into $\mathbb{R}^{n+1}$ as above. It follows that the linear sub-space $\text{Span}\{\pi^{-1}(q)\}$ spanned by $\pi^{-1}(q)$ is isometric to a $(k+1)$-dimensional $\mathbb{R}_q^{k+1}$.

For each geodesic segment of unit speed $\hat{\sigma} : [0, \frac{\pi}{2}] \to A'$ from q to $y = \hat{\sigma}(\frac{\pi}{2})$, we will show that $d_{A'}(q, y) = \frac{\pi}{2}$.

7

Let $\tilde{\sigma} : [0, \frac{\pi}{2}] \to S^{n-1}$ be a horizontal lift of $\hat{\sigma}$. As we pointed out above, we can write $\tilde{\sigma}(t) = \cos t\tilde{q} + \sin t\tilde{\sigma}'(0)$, where $\tilde{\sigma}'(0) = \tilde{y} \perp \tilde{q}$. At time $t = \frac{\pi}{2}$, the vector $\tilde{\sigma}'(\frac{\pi}{2})$ becomes horizontal. Thus, $\tilde{q} = \tilde{\sigma}'(\frac{\pi}{2})$ is orthogonal to $T_{\tilde{y}}(\pi_p^{-1}(y)) \subset R_y^{k+1}$, where $\tilde{y} = \tilde{\sigma}(\frac{\pi}{2}) \in \pi_p^{-1}(y)$. Recall that $\tilde{y} = \tilde{\sigma}'(0) \perp \tilde{q}$. Hence, $\tilde{q} \perp \mathbb{R}_y^{k+1}$.

Suppose contrary, if $d_{A'}(q, y) = \alpha < \frac{\pi}{2}$. Then there would be another length-minimizing geodesic $\hat{\sigma}_2 : [0, \alpha] \to A'$ from q to y. Using the horizontal lift $\tilde{\sigma}_2$ of $\hat{\sigma}_2$ with the initial point $\tilde{q}$, we would be able to find $\tilde{z} = \tilde{\sigma}_2(\alpha) \in \pi_p^{-1}(y)$. It would follow that the angle between $\tilde{q}$ and $\tilde{z}$ is equal to $\alpha < \frac{\pi}{2}$, which contradicts to the fact $\tilde{q} \perp \mathbb{R}_y^{k+1}$. Thus, any geodesic segment $\hat{\sigma} : [0, \frac{\pi}{2}] \to A'$ of unit speed is length-minimizing, and hence $\mathrm{Inj}_{A'}(q) = \frac{\pi}{2}$.

Let us now further prove $\mathrm{Inj}_M(q) = \frac{\pi}{2}$. Let $\sigma : [0, \frac{\pi}{2}] \to M$ be any geodesic segment with $\sigma(0) = q$ and unit speed. If $\sigma'(0) \perp A'$, then by Proposition 2.4, $z = \sigma(\frac{\pi}{2}) \in A$ and hence $d(q, \sigma(\frac{\pi}{2})) = \frac{\pi}{2}$. Thus, $\sigma : [0, \frac{\pi}{2}] \to M$ is length-minimizing in this case.

If $\sigma'(0) = (\cos \beta)\vec{v} + (\sin \beta)\vec{h}$ for some $\vec{v} \perp A'$, $\vec{h} \in T_q(A')$ and $0 < \beta < \frac{\pi}{2}$, we let $\Psi_A : [M - A'] \to A$ be the nearest point projection, and let $\Psi_{A'} : [M - A] \to A'$ the nearest point projection. Let

$$z = \sigma(\frac{\pi}{2}).$$

For $y = \Psi_{A'}(z) = \Psi_{A'}(\sigma(\frac{\pi}{2}))$, by Lemma 3.1 of [GG1], $\{q, y, z\}$ and $\Psi_{A'}(\sigma(\mathbb{R}))$ are contained in a totally geodesic 2-sphere. Moreover, $\Psi_{A'}(\sigma([0, \frac{\pi}{2}]))$ is a geodesic segment of length $\frac{\pi}{2}$. Thus, since the injectivity radius of A' is equal to $\frac{\pi}{2}$, one has that $d_{A'}(y, q)$ is equal to the length of $\Psi_{A'}(\sigma([0, \frac{\pi}{2}]))$, which is $\frac{\pi}{2}$. Let $x = \Psi_A(z) \in A$. It is clear that $d(x, q) \geq d(A, q) = \frac{\pi}{2}$. Hence, we have $\{x, y\} \subset [\partial B_{\frac{\pi}{2}}(q)] = \{q\}'$.

We already showed that $\{x, y\} \subset \{q\}'$ holds. It now follows from Proposition 1.3 of [GG1] that $\{q\}'$ is π-convex. Because z lies on a geodesic segment of length $\frac{\pi}{2} < \pi$ from x to y and $\{x, y\} \subset \{q\}'$, by the π-convexity of $\{q\}'$ we obtain that $z \in \{q\}'$. Therefore, any geodesic segment $\sigma : [0, \frac{\pi}{2}] \to M$ of unit speed from q is length-minimizing for all cases. The assertion of $\mathrm{Inj}_M(q) = \frac{\pi}{2}$ is proved.   Q.E.D.

Step 2. For the convenience to the reader, we include the detailed proof of the following elementary result.

Proposition 3.2 *Let $\tilde{\Sigma} \subset S^n$ be a $(k+1)$-dimensional submanifold which is smooth at $z \in \tilde{\Sigma}$. Suppose that $\vec{h} \in T_z(S^n)$ is a unit normal vector of $\tilde{\Sigma}$ at z, $\sigma_{z,\vec{h}}(t) = Exp_z^{S^n}(t\vec{h})$ and let $\Gamma^0_{\sigma_{z,\vec{h}}, \tilde{\Sigma}_{p,q}}$ be as in (3.1) above. Then*

(1) $\dim[\Gamma^0_{\sigma_{z,\vec{h}}, \tilde{\Sigma}_{p,q}}] = k + 1$;

(2) If $\tilde{\Sigma}$ has the first focal radius $t_0 < \frac{\pi}{2}$ along $\sigma_{z,\vec{h}}$, then there must be a non-trivial Jacobi field $\{J(t)\}$ along $\sigma_{z,\vec{h}}$ with $J'(0) = (-\cot t_0)J(0) \in T_z(\tilde{\Sigma})$, and hence $J \in \Gamma^0_{\sigma_{z,\vec{h}}, \tilde{\Sigma}_{p,q}}$.

8

Proof. (1) Let $N(\tilde{\Sigma})|_{B_\varepsilon(z)} = \{(y, \vec{w}) | y \in \tilde{\Sigma}, d(y, z) < \varepsilon, \vec{w} \perp T_y(\tilde{\Sigma})\}$ be the normal bundle of $\tilde{\Sigma}$ near z. Suppose that $G = \mathrm{Exp}^{S^n} : N(\tilde{\Sigma})|_{B_\varepsilon(z)} \to S^n$ is the exponential map of S^n restricted to the normal bundle of $\tilde{\Sigma}$. For any curve $\zeta : (-\delta, \delta) \to N(\tilde{\Sigma})$ with $\zeta(0) = (z, \vec{h})$ with $\zeta(s) = (y(s), \vec{w}(s))$, there is an 1-family of geodesics given by $F(t, s) = G(y(s), t\vec{w}(s)) = \mathrm{Exp}_{y(s)}[t\vec{w}(s)]$.

Let $J(t) = \frac{\partial F}{\partial s}(t, 0) = G_*|_{(z, t\vec{h})}\zeta'(0)$. Since $\frac{\partial F}{\partial t}(0, s) = \vec{w}(s) \perp \tilde{\Sigma}$ for all $s \in (-\delta, \delta)$, by the Gauss-Coddazi equation we obtain that, for all $X \in T_z(\tilde{\Sigma})$,

$$\langle J'(0), X \rangle = \langle \vec{w}'(0), X \rangle = -\langle \vec{w}(0), \nabla_{y'(0)} X \rangle = -\langle \vec{h}, \nabla_{J(0)} X \rangle$$

holds. Hence, the tangential component of $J'(0)$ is uniquely determined by the second fundamental form of $\tilde{\Sigma}$:

$$\langle J'(0), X \rangle = -II_{\vec{h}}(J(0), X) = -\langle \vec{h}, \nabla_{J(0)} X \rangle \tag{3.4}$$

for all $X \in T_z(\tilde{\Sigma})$. Let us consider the classical Weingarten map $W^{\vec{h}} : T_z(\tilde{\Sigma}) \to T_z(\tilde{\Sigma})$, where $W^{\vec{h}}(Y)$ is given by the second fundamental form associated with $\vec{h}$:

$$\langle W^{\vec{h}}(Y), X \rangle = -II_{\vec{h}}(Y, X) = -\langle \vec{h}, \nabla_X Y \rangle \tag{3.5}$$

for all $X \in T_z(\tilde{\Sigma})$. Hence, our Jacobi field J satisfies the Coddazzi equation

$$[J'(0)]^\top = W^{\vec{h}} J(0), \tag{3.6}$$

where $[\vec{\eta}]^\top$ denotes the tangential component of $\vec{\eta} \in T_z(S^n)$. It follows that

$$J \in \Gamma_{\sigma_{z,\vec{h}}, \tilde{\Sigma}}.$$

For $J \in \Gamma^0_{\sigma_{z,\vec{h}}, \tilde{\Sigma}}$, we further require that $J'(0) \in T_z(S^n)$. Hence, it follows from (3.6) that

$$J'(0) = W^{\vec{h}} J(0) \tag{3.7}$$

Because $J(0) \in T_z(\tilde{\Sigma})$ and $\dim(\tilde{\Sigma}) = k + 1$, by (3.7) one has $\dim[\Gamma^0_{\sigma_{z,\vec{h}}, \tilde{\Sigma}}] \leq (k + 1)$.

We now prove that $\dim[\Gamma^0_{\sigma_{z,\vec{h}}, \tilde{\Sigma}}] \geq (k + 1)$. Let $\{\vec{v}_1, ..., \vec{v}_{k+1}\}$ be an orthogonal basis of $T_z(\tilde{\Sigma})$.

It is well-known that, on each curve $s \to y_i(s)$ with $y_i(0) = z$ and $y_i'(0) = \vec{v}_i$, there is a unique a vector field $\{\vec{w}_i(s)\}$ satisfying $\vec{w}_i(s) \perp T_{y_i(s)}(\tilde{\Sigma})$ and

$$[\nabla_{y_i'(s)} \vec{w}(s)]^\perp = 0 \tag{3.8}$$

with $\vec{w}_i(0) = \vec{h}$, where $[\vec{\eta}]^\perp$ denotes the normal component of $\vec{\eta}$. The linear system (3.8) has $(n - k - 1)$-unknowns and $(n - k - 1)$-equations. Thus, the system (3.8) has a unique solution $\vec{w}_i(s)$ with $\vec{w}_i(0) = \vec{h}$.

Let $F_i(t, s) = \text{Exp}_{y_i(s)}[t\vec{w}_i(s)]$ and $J_i(t) = \frac{\partial F_i}{\partial s}(t, 0)$. Then $J_i \in \Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}}$ for $i = 1, 2, ..., k+1$. Clearly, $\{J_1(0), ..., J_{k+1}(0)\} = \{\vec{v}_1, ..., \vec{v}_{k+1}\}$ are linearly independent. Thus, $\dim[\Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}}] \geq (k+1)$.

(2) If $\tilde{\Sigma}$ has the first focal point $\sigma(t_0)$ along σ with $0 < t_0 < \frac{\pi}{2}$, then there must be an orthogonal Jacobi field $\{J(t)\}$ along σ in S^n with $J(0) \in T_z(\tilde{\Sigma})$ and $J(t_0) = 0$. It is well-known that, in S^n, any Jacobi field $\{J(t)\}$ with $J(t_0) = 0$ can be expressed as $J(t) = \sin(t - t_0)cE(t)$, where c is a non-zero constant and $\{E(t)\}$ is a unit parallel vector field along σ.

Because $0 < t_0 < \frac{\pi}{2}$, we obtain that $J(0) = -(\sin t_0)cE(0) \neq 0$. Since $J(0) \in T_z(\tilde{\Sigma})$, we see that $E(0) = -\frac{1}{\sin t_0}J(0) \in T_z(\tilde{\Sigma})$. It follows that $J'(0) = \cot(t_0)cE(t) \in \tilde{\Sigma}$ and hence $J \in \Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}}$.
$\hfill$ Q.E.D.

Step 3. We will use the Hessian comparison theorem to show that *if $\pi_p : S_p^\perp(A, M) \to A'$ is a Riemannian submersion, then $\tilde{\Sigma}_{p,q}$ has the first focal radius $\geq \frac{\pi}{2}$ in S^n, and hence $\pi_p^{-1}(q)$ is a great circle by Step 1.*

We will divide it into two sub-steps:

Step 3.1. Using the Hessian comparison theorem, we study the decomposition of $T_y(M^n)$ associated with $\{A, A'\}$ and parallel transports. As an application, we will show that *the covariant derivatives of horizontal lifting vector fields along each fiber $\tilde{\Sigma}_{p,q}$ must be vertical.*

Step 3.2. By Step 3.1, we will construct all Jacobi fields $J \in \Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}}$ with *vertical initial derivatives* explicitly. A simple calculation will show that any non-trivial element $J \in \Gamma^0_{\sigma_{z,\vec{h}},\tilde{\Sigma}}$ has the non-vanishing property $J(t) \neq 0$ for $0 \leq t < \frac{\pi}{2}$. This will complete the proof of Theorem A.

Step 3.1. Hessian comparison and the covariant derivatives of horizontal lifting vector fields along each fiber.

We will frequently use the following result.

Proposition 3.3 *(cf. [GG1]) Let A, A' and M be as in Proposition 2.4. Suppose that $(p, q) \in A \times A'$. Then*

(1) Whenever $\dim(A') > 0$, for any unit tangent vector $\vec{\eta}_0 \in T_q(A')$ and a unit normal vector $\vec{v} \in S_q^\perp(A', M)$, the image $\text{Exp}_q(\mathbb{R}^2_{\vec{\eta}_0,\vec{v}})$ is a totally geodesic immersed 2-sphere of constant sectional curvature 1, where $\mathbb{R}^2_{\vec{\eta}_0,\vec{v}} = \text{Span}_\mathbb{R}\{\vec{\eta}_0, \vec{v}\}$ is a real 2-dimensional tangent subspace spanned by $\{\vec{\eta}_0, \vec{v}\}$.

(2) Let $\Psi_{A'} : [M - A] \to A'$ be the nearest point projection, $\Phi_{A'} : ([B_{\frac{\pi}{2}}^\perp(0_p) - \{0_p\}], g_1) \to A'$ be given by $\Phi_{A'}(z) = \Psi_{A'}(\text{Exp}_p(z))$ for $z \in B_{\frac{\pi}{2}}^\perp(0_p) = \{z \in T_p(M)| z \perp T_p(A), |z| < \frac{\pi}{2}\}$ and

$z \neq 0$. Suppose that $S_{r_0}^{\perp}(0_p) = \partial B_{r_0}^{\perp}(0_p)$. Then for $r_0 \in (0, \frac{\pi}{2}) \to A'$, the map

$$\Phi_{A'}|_{S_{r_0}^{\perp}(0_p)} : S_{r_0}^{\perp}(0_p) \to A'$$

is a Riemannian submersion up to a constant factor $c = \frac{1}{\sin r_0}$ with respect to the spherical metric g_1 on $S_{r_0}^{\perp}(0_p) \subset B_{\pi}(0_p)$.

The similar conclusions hold at $p \in A$ if $\dim(A) > 0$.

Let us consider the normal bundle of $\tilde{\Sigma}_{p,q}$ at z with $0 < |z| \leq \frac{\pi}{2}$.

Definition 3.4 Let $(p, q) \in A \times A'$ be as above. If $z \in \tilde{\Sigma}_{p,q} \subset B_{\pi}(0) \subset T_p(M^n)$ with $0 < |z| \leq \frac{\pi}{2}$ and if $\vec{h} \perp T_z(\tilde{\Sigma}_{p,q})$ then the vector $\vec{h}$ is called a horizontal vector.

Similarly, if $\hat{z} \in M^n$ with $0 < d(p, \hat{z}) \leq \frac{\pi}{2}$ and $\hat{h} \perp T_z(\Sigma_{p,q})$ then the vector $\hat{h}$ is called a horizontal vector.

The horizontal subspace at $\hat{z}$ is denoted by $H_{\hat{z}}$.

We will use the Hessian comparison theorem show that the horizontal subspaces is invariant under the parallel translation along radial geodesics from A' to p. If $c : [a, b] \to M$ is a curve, we let $\tau_{c(t_1)}^{c(t_2)}$ be the parallel translation along the curve c.

Theorem 3.5 Suppose that $(p, q) \in A \times A'$ and M^n are as in Proposition 2.4 and suppose that $\sigma : [0, \frac{\pi}{2}] \to M^n$ be a geodesic of unit speed from q to p. Then the tangent space $T_{\sigma(t)}(M^n)$ has the following orthogonal decomposition:

$$T_{\sigma(t)}(M^n) = \tau_{\sigma(0)}^{\sigma(t)}[T_q(A')] \bigoplus \tau_{\sigma(\frac{\pi}{2})}^{\sigma(t)}[T_q(A)] \bigoplus T_{\sigma(t)}(\Sigma_{p,q}).$$

Hence, $\tau_{\sigma(0)}^{\sigma(t)}[T_q(A')] \bigoplus \tau_{\sigma(\frac{\pi}{2})}^{\sigma(t)}[T_q(A)]$ is equal to the horizontal subspace $H_{\sigma(t)}$ at $\sigma(t)$ for $t \in [0, \frac{\pi}{2})$.

Proof. By Lemma 3.1 of [GG1], $\sigma'(t) \perp H_{\sigma(t)}$. We need to show that $T_{\sigma(t)}(\Sigma_{p,q}) \perp H_{\sigma(t)}$. For this purpose, we use the sharp version of Hessian comparison.

Let $m = \dim A$, $m' = \dim A'$ and $k + 1 = \dim(\Sigma_{p,q})$. We will also see that $\dim M^n = n = m + m' + (k + 1)$.

Let $f(x) = d(x, A')$. Because A' is totally geodesic, there are m' Jacobi fields $\{J_1(t), J_2(t), ..., J_{m'}(t)\}$ along σ such that $\{J_1(0), J_2(0), ..., J_{m'}(0)\}$ is an orthonormal basis of $T_q(A')$ and $J_i'(0) = 0$ for $i = 1, 2, ..., m'$.

Similarly, if $\dim A > 0$, there are m Jacobi fields $\{J_{m'+1}(t), J_{m'+2}(t), ..., J_{m'+m}(t)\}$ along σ such that $\{J_{m'+1}(0), J_{m'+2}(0), ..., J_{m'+m}(0)\}$ is an orthonormal basis of $T_q(A')$ and $J_{m'+j}'(0) = 0$ for $i = 1, 2, ..., m$.

We already knew that the cut-radii of A' and A are equal to the diameter of M^n, which is $\frac{\pi}{2}$. Thus, $J_i(t) \neq 0$ for $i = 1, 2, ..., 8$ and $t \in [0, \frac{\pi}{2})$. Recall that the sectional curvature

≥ 1, by Berger comparison theorem (or the 2nd Rauch comparison theorem), we can find a parallel vector field $\{E_i(t)\}$ along σ such that $J_i(t) = \cos t E_i(t)$ for $i = 1, 2, ... m'$ and $J_{m'+j}(t) = \sin t E_{m'+j}(t)$ for $j = 1, ... m$ if $m = \dim A > 0$. It is clear that

$$Hess(f)(J, J) = \langle J(t), J'(t)\rangle.$$

It also is well-known that the Hessian of distance function f satisfies the so-called Riccati equation:
$$\nabla_{\sigma'(t)}[\text{Hess}(f)] + [\text{Hess}(f)]^2 + R = 0.$$

More precisely, we let $\{E_i(t)\}_{1 \leq i \leq n}$ be a parallel orthonormal base along the geodesic segment φ_v with $E_n(t) = \sigma'(t)$, $H_{i,j}(t) = \text{Hess}(f)(E_i(t), E_j(t))$ and $R_{ij}(t) = \langle R(\sigma(t), E_i(t))\sigma'(t), E_j(t)\rangle$, where $R(X, Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X + \nabla_{[X,Y]}Z$ is the curvature tensor. Thus, we have

$$H' + H^2 + R = 0.$$

Let
$$W_{A'}(t) = \{Y(t)| \quad H(., Y(t))|_{\sigma(t)} = \tan(t)\langle., Y(t)\rangle\}$$

and
$$W_A(t) = \{Y(t)| \quad H(., Y(t))|_{\sigma(t)} = \cot(t)\langle., Y(t)\rangle\}.$$

We have shown that the eigenspace $\{W_{A'}(t)\}$ is invariant under parallel translation along σ. Similarly, if $\dim A > 0$, then $\{W_A(t)\}$ is invariant under parallel translation along σ.

Choose $t_0 = \frac{\pi}{3}$. It is clear $\cot\frac{\pi}{3} \neq \tan\frac{\pi}{3}$. Thus,

$$W_{A'}(t) \perp W_A(t)$$

whenever $\dim A > 0$.

In what follows, we prove that

$$T_{\sigma(t)}(\Sigma_{p,q}) \perp [W_{A'}(t) \bigoplus W_A(t)].$$

We already showed that $E_j(t) \in W(t)$ for $j = 1, 2, ..., m'$. Notice that $H_{jj}(t)$ blows up as $t \to 0^+$ for $j > (m' + m)$. If $\{\lambda_{m+m'+1}(t), \lambda_{m+m'+2}, ..., \lambda_{m+m'+k}(t)\}$ are other eigenvalues of H, then $\lambda_j(t) \to +\infty$ as $t \to 0$ for $j \leq (m + m')$. Thus, the corresponding eigenvectors are orthogonal to $W_{A'}(t)$, because eigenvalues are different.

Similarly, if $\dim A > 0$, we consider $t \to \frac{\pi}{2}$, then $\lambda_j(t) \to +\infty$ as $t \to \frac{\pi}{2}$ for $j \leq (m+m')$. For the same reason, the corresponding eigenvectors are orthogonal to $W_A(t)$, because eigenvalues are different.

Therefore, we proved
$$H_{ij}(t) = 0$$

for $i = 1, ..., (m + m')$ and $j > (m + m')$.

Let $\{(x_1, .., x_{m'})\}$ be a geodesic normal coordinate system of A' at q given by $G : \mathbb{R}^{m'} \to A'$ with $G(x_1, ..., x_{m'}) = \mathrm{Exp}_q(\sum_1^{m'} x_i E_i(0))$. Recall that $\dim\{[T_q(A')]^{\perp}\} = m + k + 1$. Thus there exists an orthonomal basis $\{E_{m'+1}, ..., E_{n-1}, E_n\}$ of $[T_q(A')]^{\perp}$ such that $E_n = \sigma'(0)$. Let $\vec{\theta} = (\theta_{m'+1}, ..., \theta_n)$ with $|\vec{\theta}| \leq 1$. Then $\Psi : B_1(0) \to S^{n-m'-1} = S_q^{\perp}(A', M^n)$ given by

$$\Psi(\theta_{m'+1}, ..., \theta_{n-1}) = \sum_{j=m'+1}^{n} \theta_j E_j + \sqrt{1 - |\vec{\theta}|^2}\, \sigma'(0)$$

gives rise to a local coordinate system of $S^{n-m'-1} = S_q^{\perp}(A', M^n)$ around $\sigma'(0)$. Using the parallel transport $\tau_{G(0)}^{G(x)}$ from $q = G(0)$ to $G(x)$ we have a local coordinate system given by $(\theta_{m'+1}, ..., \theta_{n-1}) \to \tau_{G(0)}^{G(x)}(\Psi(\vec{\theta}))$ for $S_{G(x)}^{\perp}(A', M^n)$. Therefore, $\{(x_1, ..., x_{m'}; \theta_{m'+1}, ..., \theta_{n-1}, t)\}$ gives rise to a local coordinate for normal bundle of A' in M^n near $(x, t\sigma'(0))$. In fact, the $F(x_1, ...x_{m'}; \theta_{m'+1}, ..., \theta_{n-1}, t) = \mathrm{Exp}_{G(x)}[t\tau_{G(0)}^{G(x)}(\Psi(\vec{\theta}))]$ does the job. Finally we let $C_{ji}(t) = \langle \frac{\partial F}{\partial \theta_j}, E_i \rangle|_{\sigma(t)}$. It is well-known that $H(t) = C'(t)[C(t)]^{-1}$. We already showed that $W_{A'}(0) = T_q(A')$ and $\{W(t)\}$ is parallel along σ. Using $C_{ji}(0) = 0$ and the fact $H_{ij}(t) = 0$ for $i = 1, ..., m'$ and $j > (m' + 1)$, by the integration of $C'(t) = H(t)C(t)$ from $\frac{\pi}{2}$ to t we conclude that

$$C_{ij}(t) = 0$$

for $i = 1, ..., m'$ and $j > (m' + 1)$. Thus, we see that $\frac{\partial F}{\partial \theta_j} \in [W_{A'}(t)]^{\perp}$ for $j > (m' + 1)$.

Therefore, both tangential subspace $T_{\sigma(t)}(\Sigma_{p,q}) \bigoplus W_A(t)$ and sub-space $W_{A'}(t)$ at $\sigma(t)$ are invariant under parallel translation along σ. It follows that

$$T_{\sigma(t)}(\Sigma_{p,q}) \perp W_{A'}(t).$$

For the same reason, if $\dim A > 0$, one has

$$T_{\sigma(t)}(\Sigma_{p,q}) \perp W_A(t)$$

as well. We already proved $W_A(t) \perp W_{A'}(t)$. This completes the proof.   Q.E.D.

Theorem 3.5 indicates that there is a non-trivial relation between the exponential map Exp_A along the normal bundle of A and the exponential map $\mathrm{Exp}_{A'}$ along the normal bundle of A'. As an application of Theorem 3.5, we draw some conclusions.

Corollary 3.6 *Let $(p, q) \in A \times A'$, A and A' be as in Proposition 2.4 and $\dim A' > 0$. Suppose that $\vec{\eta} \in T_q(A')$, $\hat{z} \in \Sigma_{p,q}$ with $0 < d(\hat{z}, A') < \frac{\pi}{2}$, $\hat{h}_{\eta}(\hat{z})$ is the parallel transport of $\vec{\eta}$ along the unique length-minimizing geodesic segment from q to $\hat{z}$, $z = (Exp_p)^{-1}(\hat{z}) \in \hat{\Sigma}_{p,q}$ and*

$$\vec{h}_{\eta}(z) = [(Exp_p)_*^{-1}]|_z \hat{h}_{\eta}(\hat{z}).$$

Then the horizontal lifting vector field $\{\vec{h}_\eta\}_{z \in \tilde{\Sigma}_{p,q}}$ *of* η *has the property*

$$\nabla_X \vec{h}_\eta \in T_z(\tilde{\Sigma}_{p,q}) \tag{3.9}$$

for all $X \in T_z(\tilde{\Sigma}_{p,q})$.

Proof. We first consider the case $X = \nabla r$, where $r(z) = |z| = d(0_p, z)$. By our assumption, there is a unique geodesic segment of unit speed from q to $\hat{z}$, say $\sigma_{q,\hat{z}}$. Let $\vec{v} = \sigma'_{q,\hat{z}}(0)$. By Proposition 3.3, if we let $\mathbb{R}^2_{\vec{\eta},\vec{v}}$ be the subspace spanned by $\{\vec{\eta}, \vec{v}\}$, then $\hat{S}^2 = \mathrm{Exp}_q(\mathbb{R}^2_{\vec{\eta},\vec{v}})$ is a totally geodesic immersed 2-sphere $S^2_{\vec{\eta},\vec{v}}$ of constant curvature 1, which passes both p and q. It follows that, on the unit 2-sphere $S^2_{\vec{\eta},\vec{v}}$, on has

$$\nabla_{\nabla r} \vec{h}_\eta |_z = 0 \tag{3.10}$$

We now consider the remaining case $X \in T_z(\tilde{\Sigma}_{p,q})$ but $X \perp \nabla r$. Let

$$B^\perp_{0_p, \pi} = \{z \in T_p(M) \,|\, z \perp T_p(A), |z| \leq \pi\}.$$

In terms of the spherical metric g_1, the sub-manifold $(B^\perp_{p,\pi}, g_1)$ is a totally geodesic $(n - m)$-dimensional sphere S^{n-m}, where $m = \dim A$, $m' = \dim A'$, $k + 1 = \dim \Sigma_{p,q}$ and $n = \dim M = m + m' + k + 1$.

Let $\Psi_{A'} : [M - A] \to A'$ be the nearest point projection, $S^\perp_{p,r} = \{z \in T_p(M) \,|\, z \perp T_p(A), |z| = r\}$. By Proposition 3.3 and Theorem 3.5, in terms of the spherical metric g_1 on $B_\pi(0_p)$, the map

$$\tilde{\Psi}_{A'} : \quad S^\perp_{p,r} \quad \to \quad A'$$
$$z \quad \to \quad \Psi_{A'}[\mathrm{Exp}_p(z)]$$

is a Riemannian submersion up to a constant factor $\frac{1}{\sin r}$. Consequently, if $\{\vec{\eta}_1, ..., \vec{\eta}_{m'}\}$ is an orthonormal basis of $T_q(A')$, then by Theorem 3.5 and its proof, the set of vectors

$$\{\vec{h}_{\vec{\eta}_1}, ..., \vec{h}_{\vec{\eta}_{m'}}\}$$

form a basis of the normal bundle $N(\tilde{\Sigma}_{p,q}, B^\perp_{0_p, \pi})$ of $\tilde{\Sigma}_{p,q}$ at z in $S^{n-m} = (B^\perp_{0_p, \pi}, g_1)$.

Let $\{x_1, ..., x_{m'}\}$ be the geodesic normal coordinate of A' at q. We choose $\vec{\eta}_i = \frac{\partial}{\partial x_i}$ at 0_q; i.e., we use the map $(x_1, ..., x_{m'}) \to \mathrm{Exp}_q(x_1 \vec{\eta}_1 + ... x_{m'} \vec{\eta}_{m'})$ as the geodesic coordinate system of A' at q.

Suppose that $G(x) = \mathrm{Exp}_q(x_1 \vec{\eta}_1 + ... x_{m'} \vec{\eta}_{m'})$ and recall that $\vec{v} = \sigma'_{q,\hat{z}}(0)$. Let us consider the Fermi coordinate system (the exponential map) along A':

$$F(x, \rho \vec{v}) = \mathrm{Exp}_{G(x)}[\tau_q^{G(x)} \rho \vec{v}].$$

14

By the proof of Theorem 3.5, we have

$$\vec{h}_{\vec{\eta}_i} = \frac{1}{\sin|z|}[\frac{\partial(\text{Exp}_p^{-1} \circ F)}{\partial x_i}]|_{(0_q,\rho_0\vec{v})}, \tag{3.11}$$

where

$$\rho_0 = \frac{\pi}{2} - |z|.$$

For simplicity, we denote $\text{Exp}_p^{-1} \circ F$ by $\tilde{F}$. We now choose $X = \frac{\partial \tilde{F}}{\partial v_i}|_{(0_q,\rho_0\vec{v})}$ for $\vec{v} = (v_1,...,v_k) \in [T_q(A')]^\perp$ and $|\vec{v}| = 1$, where we only allow $\vec{v} \in S_q^\perp(A',M)$. It is easy to see (cf. [CE, page2]) that, if $[X,Y] = 0 = [Y,Z] = [X,Z]$, then

$$\langle \nabla_X Y, Z \rangle = X\langle Y,Z \rangle + Y\langle X,Z \rangle - Z\langle X,Y \rangle.$$

Therefore, setting $\rho_0 = \frac{\pi}{2} - |z|$ and by a direct calculation, one has that, if $X = \frac{\partial \tilde{F}}{\partial v_i}|_{(0_q,\rho_0\vec{v})}$ then

$$\langle \nabla_X \vec{h}_{\vec{\eta}_i}, \vec{h}_{\vec{\eta}_j} \rangle = 0 + 0 - 0 = 0$$

for all $i,j = 1,...,m'$. This completes the proof. $\hspace{2cm}$ Q.E.D.

A direct consequence of the above corollary is the following result.

Corollary 3.7 *Let $(p,q) \in A \times A'$, A and A' be as in Proposition 2.4 and $\dim A' > 0$. $\vec{\eta} \in T_q(A')$, $\hat{z} \in \Sigma_{p,q}$ with $0 < d(\hat{z}, A') < \frac{\pi}{2}$, $\hat{h}_\eta(\hat{z})$ is the parallel transport of $\vec{\eta}$ along the unique length-minimizing geodesic segment from q to $\hat{z}$, $z = (Exp_p)^{-1}(\hat{z}) \in \tilde{\Sigma}_{p,q}$,*

$$\vec{h}_\eta(z) = [(Exp_p)_*^{-1}]|_z \hat{h}_\eta(\hat{z})$$

and

$$F_{\vec{\eta}}(t,z) = Exp_z^{S^n}[t\vec{h}_\eta(z)]. \tag{3.12}$$

Then the corresponding Jacobi fields

$$J_i(t) = \frac{\partial F_{\vec{\eta}}}{\partial z_i}(t,z) \tag{3.13}$$

along the geodesic $\{F_{\vec{\eta}}(.,z)\}$ has the property

$$J_i'(0) \in T_z(\tilde{\Sigma}_{p,q}) \tag{3.14}$$

for $i = 1,...,k+1$, where $\{z_1,....,z_{k+1})\}$ is any local coordinate system of $\tilde{\Sigma}_{p,q}$ around z.

Step 3.2. Proof of Theorem A.

We recall some elementary facts about the geodesic triangles in a unit 2-sphere S^2, which are isometrically immersed in M, see Proposition 3.3 (1) above.

Lemma 3.1 *Let $\hat{z} \in \Sigma_{p,q}$ with $0 < r_0 = d(\hat{z}, A) < \frac{\pi}{2}$ and $S^2 = S^2_{q,(z,\hat{h})}$ be a totally geodesic immersed 2-sphere in M given by $S^2_{q,(z,\hat{h})} = Exp_{\hat{z}}(\mathbb{R}^2_{q,(z,\hat{h})})$, where $\hat{h}$ is a unit horizontal vector and $\mathbb{R}^2_{q,(z,\hat{h})} = Span\{\hat{h}, Exp_{\hat{z}}^{-1}(q)\}$ described as in Proposition 3.3(1).*

(1) If $\varphi_{\hat{h}}(t) = Exp_{\hat{z}}(t\hat{h})$ for some unit horizontal vector at $\hat{z}$ with $0 < r_0 = d(\hat{z}, A) < \frac{\pi}{2}$, then $\varphi_{\hat{h}}(\frac{\pi}{2}) \in A'$;

(2) For $0 < t < \frac{\pi}{2}$, the distance function satisfies $r(t) = d(\varphi_{\hat{h}}(t), A) = \arccos[\cos r_0 \cos t]$; Consequently, $d(\varphi_{\hat{h}}(t), A') = \arcsin[\cos r_0 \cos t]$, where $r_0 = d(\hat{z}, A)$.

(3) Let $\Psi_{A'} : [M - A] \to A'$ be the nearest point projection and $\ell(t)$ be the length of $\Psi_{A'}[\varphi_{\hat{h}}([0, t])]$. Then

$$\ell(t) = \arccos\left[\frac{\sin r_0 \cos t}{\sqrt{1 - (\cos r_0 \cos t)^2}}\right].$$

(4) The vector $[\varphi_{\hat{h}}'(t) - \langle \varphi_{\hat{h}}'(t), \nabla r \rangle \nabla r]$ remains to be horizontal, where $r(x) = d(A, x)$.

The lemma above can be proved by the law of cosine in S^2, see [Pe, page 314].
Finally, we can now show that $\tilde{\Sigma}_{p,q}$ has focal radius $\geq \frac{\pi}{2}$ in S^n.

Lemma 3.2 *Let $z \in \tilde{\Sigma}_{p,q} \subset S^n$ and $J_i(t)$ be as in Corollary 3.7 above. Then*

$$J_i(t) \neq 0$$

for all $t \in (0, \frac{\pi}{2})$. Consequently, $\tilde{\Sigma}_{p,q}$ has focal radius $\geq \frac{\pi}{2}$ in S^n.

Proof. We choose a special local coordinate system of $\tilde{\Sigma}_{p,q}$ at z as follows. By Corollary 3.7, $J_i'(0) \in T_z(\tilde{\Sigma}_{p,q})$ for all $i = 1, ..., k+1$. We can choose $(k+1)$-principal directions $\{e_1, ..., e_{k+1}\}$ of the Weingart map $W^{\vec{h}} : X \to (\nabla_X \vec{h}(z))^\top = \nabla_X \vec{h}(z)$ for all $X \in T_z(M)$, where

$$\langle W^{\vec{h}} X, Y \rangle = \langle \nabla_X \vec{h}(z), Y \rangle$$

for all $X, Y \in T_z(M)$ and $(\vec{w})^\top$ is the tangential component of $\vec{w}$.

It was proved $\nabla r|_z = \frac{z}{|z|}$ and $\vec{h}(z)$ span a totally geodesic 2-sphere of constant curvature 1, see Proposition 3.3 (1) above. Thus, $\nabla r|_z$ is an eigenvector of $W^{\vec{h}}$. We choose $e_{k+1} = \nabla r|_z$. Furthermore, in S^2, the corresponding Jacobi field can be written as $J_{k+1}(t) = (\cos t)E_{k+1}(t)$, where $\{E(t)\}$ is a parallel vector along $\sigma_{z,\vec{h}}(t) = Exp_z(t\vec{h})$ with $E(0) = \nabla r|_z$. Hence, $J_{k+1}(t) \neq 0$ for all $t \in (0, \frac{\pi}{2})$.

We now consider the remaining $\{J_1, ..., J_k\}$. Let

$$v_i(s) = |z|[(\cos s)\frac{z}{|z|} + (\sin s)e_i]$$

and

$$F_i(t, s) = E^{S^n}_{v_i(s)}(t\vec{h}_{\vec{\eta}}(v_i(s)))$$

for $i = 1, ..., k$. Finally, we set

$$J_i(t) = \frac{\partial F}{\partial s}(t, 0)$$

for $i = 1, ..., k$.

In order to prove that $J_i(t) \neq 0$ for $t \in (0, \frac{\pi}{2})$, we use

$$\hat{F}_i(t, s) = \mathrm{Exp}^M_p[(\mathrm{Exp}^{S^n}_p)^{-1}(F_i(t, s))]$$

for $i = 1, ..., k$. By Lemma 3.1, one has

$$0 < \rho(t) = d(A', \hat{F}_i(t, s)) = \arcsin[(\cos |z|) \cos t] < \frac{\pi}{2} \tag{3.15}$$

for $0 \leq t < \frac{\pi}{2}$. Let $G = \mathrm{Exp}^M_p[(\mathrm{Exp}^{S^n}_p)^{-1}]$ and

$$\hat{J}_i(t) = \frac{\partial \hat{F}_i}{\partial s}(t, 0) = G_* J_i(t).$$

Because G is a local diffeomorphism at all $x \in B_{\frac{\pi}{2}}(0_p)$ with $0 < |x| < \frac{\pi}{2}$, using (3.15) one concludes the following is true: "$J_i(t) \neq 0$ holds for $t \in (0, \frac{\pi}{2})$ if and only if $\hat{J}_i(t) \neq 0$ holds for $t \in (0, \frac{\pi}{2})$".

It remains to verify that $\hat{J}_i(t) \neq 0$ for $t \in (0, \frac{\pi}{2})$. For this purpose, we express $\hat{J}_i(t)$ in terms of the Fermi coordinates along A' instead. In terms of the Fermi coordinates along A', we will clearly see that $\hat{J}_i(t) \neq 0$ for $t \in (0, \frac{\pi}{2})$. The detail for the new expressions of $\hat{J}_i(t)$ and $\hat{F}_i(t, s)$ can be given as follows:

Notice that $\{\hat{h}_{\vec{\eta}}(v_i(s)), \mathrm{Exp}^{-1}_{v_i(s)}(q)\}$ span a totally geodesic immersed 2-sphere $S^2_{v_i(s), \hat{h}_{\vec{\eta}}}$ of constant curvature 1. Such a 2-sphere $S^2_{v_i(s), \hat{h}_{\vec{\eta}}}$ passes through the geodesic $\hat{\sigma}_{\vec{\eta}}(\ell) = \mathrm{Exp}_q(\ell\vec{\eta})$. Let

$$\vec{\psi}_i(s) = \mathrm{Exp}^{-1}_q[\hat{F}_i(0, s)], \tag{3.16}$$

$\tau^{\hat{\sigma}(\ell)}_q$ be the parallel translation along $\hat{\sigma}_{\vec{\eta}}$ and let $\Psi_{A'} : [M - A] \to A'$ be the nearest point projection. Then, by Lemma 3.1, one has

$$\ell(t) = d(\Psi_{A'}(\varphi_{\hat{h}}(t), q) = \arccos[\frac{\sin r_0 \cos t}{\sqrt{1 - (\cos r_0 \cos t)^2}}].$$

A direct calculation shows that if $q(t) = \hat{\sigma}_{\vec{\eta}}(\ell(t))$ then

$$\hat{F}_i(t,s) = \mathrm{Exp}_{q(t)}[\tau_q^{q(t)}\rho(t)\vec{\psi}_i(s)] \tag{3.17}$$

for $i = 1, ..., k$. It follows that

$$\hat{J}_i(t) = [\mathrm{Exp}_{q(t)}]_*[\tau_q^{q(t)}(\rho(t)\vec{\psi}'_i(0))] \tag{3.18}$$

We already proved that $0 < \rho(t) < \frac{\pi}{2}$. Recall that A' is totally geodesic and

$$[\tau_q^{q(t)}(\rho(t)\vec{\psi}'_i(0))] \perp T_{q(t)}(A') \tag{3.19}$$

for all t. Recall that the parallel transport $\tau_q^{q(t)} : T_q(M) \to T_{q(t)}(M)$ is an isometry. Since the cut radius of A' is equal to $\frac{\pi}{2}$, it follows equations (3.15) -(3.19) that

$$\hat{J}_i(t) = [\mathrm{Exp}_{q(t)}]_*[\tau_q^{q(t)}\rho(t)\vec{\psi}'_i(0)] \neq 0$$

for $i = 1, ..., k$ and $t \in (0, \frac{\pi}{2})$, as long as $\vec{\psi}'_i(0) \neq 0$ and $\rho(t) \neq 0$. Recall that $J_i(0) \neq 0$ and $\rho(t) \neq 0$ for $t \in (0, \frac{\pi}{2})$. Hence $\vec{\psi}'_i(0) \neq 0$ and $\hat{J}_i(t) \neq 0$ holds for $i = 1, ..., k$ and $t \in (0, \frac{\pi}{2})$. This completes the proof. Q.E.D.

The end of the proof of Theorem A. By Steps 1-3 above, we proved that $\pi_p^{-1}(q)$ is a great circle for each $(p, q) \in A \times A'$. Furthermore, it follows from Proposition 3.1(2) that $\mathrm{Diam}(A') = \frac{\pi}{2}$. We can also choose a point $y \in A'$ with $d(y, q) = \frac{\pi}{2}$. Using Proposition 3.1(2) again, we see that $\mathrm{Inj}_M(y) = \frac{\pi}{2}$. By replacing p by y if needed, we may always assume that $\mathrm{Inj}_M(p) = \frac{\pi}{2}$ and $\dim A = 0$. Hence, by [Ran], $\pi_p : S_p(M) \to A'$ is isometric to the classical Hopf fibration and M is isometric to one of $\{\mathbb{C}P^{\frac{n}{2}}, \mathbb{H}P^{\frac{n}{4}}, \mathbb{C}aP^2\}$.

Professor Grove kindly pointed out that "if $\pi_p : S_p(M) \to A'$ is a great circle fibration then one can show that M^n is isometric to one of $\{\mathbb{C}P^{\frac{n}{2}}, \mathbb{H}P^{\frac{n}{4}}, \mathbb{C}aP^2\}$ directly without using [Ran]." The following argument is an outline of a direct proof inspired by Professor Grove, but authors are responsible for all possible errors.

Let $y \in A'$ with $d(y, q) = \frac{\pi}{2}$ be as above and M' be the convex hull of $\{y\} \cup \Sigma_{p,q}$ in M. Then, by the π-convexity of $S_{\frac{\pi}{2}}(y)$ described in [GG1], one has $\pi_y : S_y(M') \to \Sigma_{p,q}$ a Riemannian submersion as well. Steps 1-3 above implies that $\pi_y : S_y(M') \to \Sigma_{p,q}$ is a great circle fibration. For any great circle fibration $\pi_y : S_y(M') \to \Sigma_{p,q}$, using O'Neill formula, one can easily show that $\Sigma_{p,q}$ is isometric to a round sphere of constant curvature 4. Thus, each fiber $\Sigma_{p,q}$ is isometric to S^{k+1} up to a factor $\frac{1}{2}$.

The metric of (M^n, g) can now be explicitly expressed as follows.

Recall that $M = \cup_{q \in A'}\Sigma_{p,q}$. For each $\hat{z} \in M^n$ and $\xi \in T_{\hat{z}}(M)$ with $r(\hat{z}) = d(p, \hat{z})$, we let ξ^H denote the horizontal component of ξ and we let ξ^v denote the vertical component of ξ.

18

Since each $\Sigma_{p,q}$ is isometric to S^{k+1} up to a factor $\frac{1}{2}$ and $\pi_p : S_p(M) \to A'$ is a great circle fibration, we have

$$|\xi|_g^2 = (\sin r)^2 |\xi^H|^2 + [\frac{1}{2}\sin(2r)]^2 |\xi^v|^2. \tag{3.20}$$

Using (3.20) and an induction method on $\frac{\dim M}{k}$, one can show that (M, g) is isometric to one of $\{\mathbb{C}P^{\frac{n}{2}}, \mathbb{H}P^{\frac{n}{4}}, \mathbb{C}aP^2\}$. Q.E.D.

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